Application of Moment Expansion Method to Options Square Root Model

Project Proposal Statement for AMSC 663

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Abstract

We implement the moment expansion based solution to the Options Square Model and we compare it to the Fourier Transform based solution. The stochastic volatility model developed by Heston (1993) is used as the Options Square Root Model or Heston Model. The governing equations consider not only the stochastic spot return but also stochastic volatility, which has a correlation with spot return. Heston (1993) also gave a closed-form solution for the European Call option price based on Fourier Transform. Different from the Fourier Transform approach, we use moment expansion. The moment generating function is used to derive 1 to at least 6 order moments to calculate the options price. This moment expansion based solution is compared with Fourier Transform based solution in terms of accuracy, and implementation difficulty.

1 Background

Because it is easy to calculate and explicitly models the relationship of all the variables, the Black-Scholes Model has been widely and successfully used in explaining stock option prices. However, the strong assumption in Black-Scholes Model that stock returns are normally distributed with constant variance and mean is not true in reality. Emprical study shows that in reality security prices do not follow a strict stationary log-normal process and the variance is non-constant. Starting from this point, some following work like Hull and White (1987) proposed a new model with stochastic volatility. These types of models could not provide a closed form solutions and involve more numerical techniques. Heston(1993) proposed a new stochastic volatility model describing the evolution of volatility of the underlying asset and also provided a closed-form solution. The basic Heston model assumes that S_t, the price of the asset, is determined by a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^s \qquad (1)$$

where ν_t, the instantaneous variance, is a CIR (Cox-Ingersoll-Ross) process:

$$dv_t = k(\theta - v_t)dt + \xi\sqrt{v_t} dW_t^v \qquad (2)$$

$$dW_t^s dW_t^v = \rho dt$$

and dW_t^s, dW_t^v are Wiener Process with correlation ρ.

The parameters in the above equations represent the following:

- μ is the average rate of return of the asset.
- θ is the long vol, or long run average price volatility; as t tends to infinity, the expected value of ν_t tends to θ.
- κ is the rate at which ν_t reverts to θ.
- ξ is the vol of vol, or volatility of the volatility, i.e, the variance of ν_t.

The Wiener Process W_t is characterized by three facts:

1. $W_0 = 0$
2. W_t is almost surely continuous
3. W_t has independent increments with distribution $W_t - W_s \sim N(0, t-s)$ (for $0 \le s < t$).

$N(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2. The condition that it has independent increments means that if $0 \le s_1 \le t_1 \le s_2 \le t_2$ then $W_{t1} - W_{s1}$ and $W_{t2} - W_{s2}$ are independent random variables, and the similar condition holds for n increments. An alternative characterization of the Wiener Process is an almost surely continuous martingale with $W_0 = 0$ and quadratic variation $[W_t, W_t] = t$ (which means that W_t^2-t is also a martingale).

The CIR process is a Markov process with continuous paths defined by the following stochastic differential equation:

$$dr_t = -\theta(r_t - \mu)dt + \sigma\sqrt{r_t} dW_t$$

where θ and σ are parameters. Value r_t follows a noncentral Chi-Square distribution. The CIR process is widely used to model short term interest rate.

To solve for option pricing, we need to figure out how to determine the option price. Consider a European call option which can be only exercised at the expiration date has a payoff $(S(T)-K)^+$, where $S(T)$ is the asset price at the expiration time T and K is the strike price. We denote $c(t,S(t))$ as the option price at time t. Then we have the termination condition:

$C(T,S(T))=(S(T)-K)^+$

We can also get two boundary conditions:

1) $c(t,0)=0$ for all $t\in[0,T]$
2) $\lim_{s(T)\to\infty}[c(t,S(t))-(S(t)-e^{-r(T-t)}K)]=0$ for all $t\in[0,T]$

We will use these two boundary conditions to solve the problems.

2 Approach

In order to use the moment generating function, we need to get first order to at least sixth order moment. Compared with the Fourier transform based solution, we can determine the maximum moment order to achieve a high accuracy solution. Here, we go through the derivation of the first moment.

Let $x=\ln S(t)$, the spot return, according to equation (1), we can have

$dx=(\mu-\frac{1}{2}v)dt+\sqrt{v}dW_t^s$

At the expiration time $M(x,v,T)=x^n$.

Then, with equation (2) $dv=k(\theta-v)dt+\xi\sqrt{v}dW_t^v$ to formulate the Kolmogorov Backward Equation:

$$\tfrac{1}{2}vM_{xx}+\rho\xi vM_{xv}+\tfrac{1}{2}\xi^2vM_{vv}+(\mu-\tfrac{1}{2}v)M_x+k(\theta-v)M_v=M_\tau \tag{3}$$

Guess $M(x,v,\tau,n)=\sum_{i=0}^{n}\sum_{j=0}^{n-i}C_{ij}{}^{n}(\tau)x^iv^j$, when considering n=1, then

$$M(x,v,\tau,1)=C_{10}(\tau)x+C_{01}(\tau)v+C_{00}(\tau) \tag{4}$$

Substitute equation (4) into (3), we have

$(\mu-\frac{1}{2}v)C_{10}(\tau)+k(\theta-v)C_{01}(\tau)=C_{10}'(\tau)x+C_{01}'(\tau)v+C_{00}'(\tau)$

Then we can get two ordinary differential equations,

$$-(\tfrac{1}{2}C_{10}(\tau)+kC_{01}(\tau))v=C_{01}'(\tau)v \tag{5}$$

$$\mu C_{10}(\tau)+k\theta C_{01}(\tau)=C_{00}'(\tau) \tag{6}$$

Since $M(x,v,\tau,1)=x$, we can easily get $C_{10}(\tau)=1$.

Solve equation (5) and (6), we get

$C_{01}=A\mathrm{e}^{-k\tau}-\dfrac{1}{2k}$ and $C_{00}=-\theta Ae^{-k\tau}+(\mu-\dfrac{\theta}{2})\tau+B$, where A and B are constants.

Similarly, we can get nth order moment writing into the polynomial equations.

To determine which nth moment is enough to satisfy accuracy requirement, we will use the Fourier transform solution as truth and compare the relative error. Therefore, we can use the moment expansion solutions as an approximation of the Fourier transform solution.

Heston (1993) guessed a solution to the Heston model, which involves two parts, one is the present value of the spot asset before optimal exercise, and the other is the present value of the strike-price payment. The solution has the following form:

$C(s, v, t) = SP_1 - KP(t, T)P_2$

Where $P(t,T) = \mathrm{e}^{-(T-t)}$ is the price at time t of a unit discount bond that matures at time T. Both of these two terms satisfy equation (3).

$P_{1,}P_2$ satisfy the terminal condition,

$P_j(x, v, T; \ln[K]) = 1_{(x \geq \ln[K])}$.

and have characteristic functions $f_j(x, v, t; \phi)$ respectively which also satisfy equation (3). $P_{1,}P_2$ can be obtained from the solutions of these characteristic functions. Then the call option prices can be obtained.

To check out the accuracy, we compare the Fourier Transform based solutions C_F with 1st to nth order moment expansion based solutions C_M^n. The graph of $\| C_M^n - C_F \|$ and n could help us to determine a cutoff and find the good enough estimation.

3 Implementation

I will use Mathematica to derive nth order moment equations. The Square Root Model with the moment expansion based solutions will be coded in Matlab 7.5. The simulation and testing part will also be done in Matlab 7.5. The computation will be done on mobile platform (laptop) or on GRACE (Glue Research and Academic Computing Environment) which consists of five Sun servers. Two of them have four 1.6 GHz UltraSPARC IIIi processors, the other two have four 2.4 GHz Opteron processors and the Oracle server has dual 1.5 GHz UltraSPACE IIIi processors. More information on www.grace.umd.edu.

4 Validation/Testing

The basic validation is to reduce volatility v as constant, i.e, $v_t = \theta$ and $\xi = 0$. The Heston model is reduced to the Black-Scholes Model, and the model should give the solution same as Black-Scholes Model solution.

We will use estimated parameters in previous literature. The idea to estimate parameter is as following. First, use the Heston model solutions to get option prices with fitting stock prices, strike prices and interest rate. Then use this to retrieve the corresponding Black-Scholes model implied volatilities $\sigma_i(\kappa, v, \xi, \rho, \theta)$. Next is to define an objective function, which is the sum of squared errors (SSE) here:

$SSE(\kappa, v, \xi, \rho, \theta) = \sum_{i=1}^{n} \{\sigma_i^{'} - \sigma_i(\kappa, v, \xi, \rho, \theta)\}^2$ where $\sigma_i^{'}$ is the observation.

Finally, minimizing this object function could find the optimal set of parameters.

5 Project Schedule

2008

October
Presentation on Project Proposal
Finish Project Proposal
Derive the first to sixth order moment equations in Mathematica

November
Complete the moment derivation
Compare the moment expansion solution with Fourier Transform solution to estimate the nth moment needed

December
Use Square Root Model with Fourier Transform based solutions to do data testing
Midyear presentation
Finish Midyear Project report

2009

January
Complete coding the Square Root Model with moment expansion solutions in Matlab
Implement the new solutions to get option prices
Consider some data testing on this new Model

February
Compare the Square Root Model with moment expansion based solutions with the one with Fourier Transform based solutions to determine a good nth moment
Use the same WRDS data to test the two models
Do statistical analysis on the two results.

March
Begin final project report write-up.

April
Complete final draft of report including edits from instructor and advisor.

May
Present final report

6 Bibliography

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